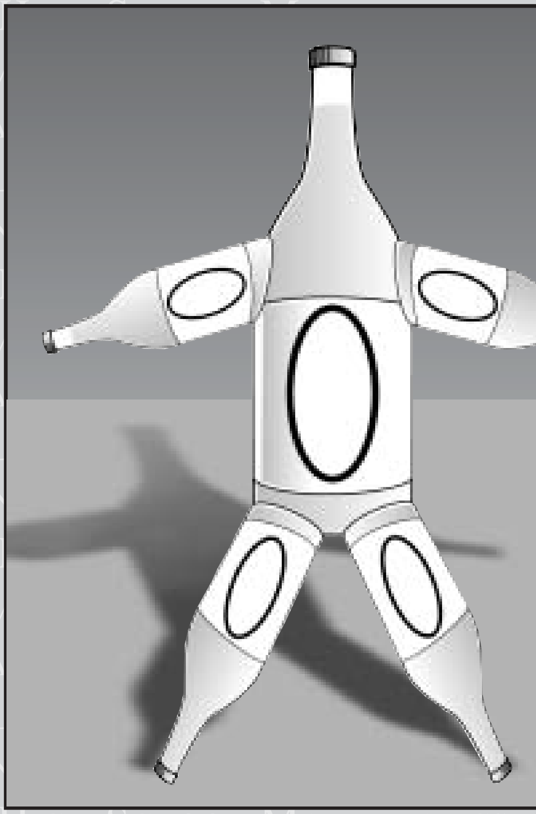
CONSORTIUM 66

The HiMAP Pull-Out Section



Alcohol and Your Body

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This lesson presents a mathematical model of how alcohol is eliminated from the body. The mathematical focus is on graphs of rational functions with an emphasis on interpretation of horizontal and vertical asymptotes in the context of elimination of alcohol from the body. Other mathematics involved is algebraic manipulation of the rational functions, solution of equations with rational expressions, realistic domain of a function, inverse functions, and equilibrium state of a dynamic process.

The Biology

The body deals with chemicals in the bloodstream primarily in two ways: the elimination by the kidneys, and the breakdown of the chemicals by enzymes from the liver. When the kidneys eliminate a chemical, they tend to eliminate a certain proportion each time period. For example, the average person eliminates about 13% of the caffeine in his or her body each hour. The liver eliminates chemicals by breaking them down with enzymes; however, the liver may not break down a constant proportion each hour. Instead, the percent of the chemical being broken down can depend on the amount of the chemical in the body; this is the case for alcohol. What happens is that as the amount of alcohol in the body increases, the proportion of the alcohol the body eliminates decreases. For alcohol, the **proportion** p of the alcohol broken down in a given hour is approximated by the formula $p = \frac{10}{4.2 + a}$, where a is the number of grams of alcohol in the body at the beginning of the hour. This is an example of what is called **capacity-limited metabolism,** in which the amount of the chemical metabolized depends on the amount of the chemical in the body. (For any individual, the numbers in the formula, 10 and 4.2, may vary considerably.)

Caffeine and Alcohol: Different Biochemistry, Different Mathematics

When caffeine enters the bloodstream, the body eliminates it at a rate proportional to the amount present, no matter how much is present. The proportion of alcohol that the body can eliminate depends upon how much is present. With alcohol, the more there is, the smaller the fraction of it that can be eliminated in a given time period.

For a 150 lb person, 34 grams of alcohol in the bloodstream produces intoxication at the level of impairment, including a charge of DWI, driving while intoxicated.

210 grams of alcohol in the bloodstream places the drinker at risk of a coma and the possibility of death.

You Try It # 1

Find the proportion of alcohol eliminated from the body during the next hour if there are 14, 28, and 42 grams, respectively, of alcohol in the body at the beginning of the hour. (One drink consists of approximately 14 grams of alcohol.)

You Try It # 2

The function $p = \frac{10}{4.2 + a}$ is only approximate. Further, it can be used only for values of a within a certain range. If a person weighing 150 pounds (about 70 kg) consumes more than 210 grams of alcohol, the person could die; thus, the body would no longer eliminate alcohol. As another example, if $a = 0.8$ grams, then $p = 2$, meaning the proportion of alcohol eliminated would be 200%. But the body cannot eliminate more alcohol than there is. For what amount of alcohol would the body eliminate 100%? (The formula is only useful for values of a above this number.)

You Try It # 3

Graph the function p. What does the graph say about the proportion of alcohol eliminated when a, the amount of alcohol in the body, is relatively large? How does this relate to the horizontal asymptote for this function? Use TRACE to estimate for which values of a the body would eliminate less than 10% of the alcohol per hour.

The Process of Elimination

To find the **amount** of alcohol eliminated in an hour, we would multiply the proportion eliminated times the amount in the body at the beginning of the hour, giving the formula, $y = \frac{10a}{4.2 + a}$, where y is the total amount of alcohol eliminated during an hour if a is the amount of alcohol in the body at the beginning of the hour.

You Try It # 4

Although the proportion of alcohol eliminated decreases, the actual amount of alcohol eliminated increases. How much alcohol is eliminated during an hour if the body began with 14, 28, and 42 grams? Compare your results to the results of You Try It # 1.

You Try It # 5

Graph the function and find its horizontal asymptote. What does this asymptote say about the amount of alcohol eliminated from the body for someone with a large amount of alcohol already in the body?

You Try It # 6

A person weighing 150 pounds with 210 grams of alcohol in the body is at risk of going into a coma or even dying. The same person with 34 grams in the body is considered legally intoxicated in many states and should not drive. Use your knowledge of the asymptote to the function $y = \frac{10a}{4.2 + a}$ to **estimate** how long this person should wait before driving if he or she has 160 grams of alcohol in the body. Is this estimate an underestimate or overestimate? Why? (Hint: Look at the graph and the asymptote.)

That this function has a horizontal asymptote implies that for some chemicals, such as alcohol, the body eliminates a virtually **constant amount** each hour when the levels, a, are fairly large. This is in contrast to other chemicals such as caffeine, of which the body eliminates a **constant proportion** each hour.

Equilibrium

Suppose a person continues to drink, ingesting 14 grams of alcohol (one drink) each hour. The change in the amount of alcohol in this person's body is the amount ingested minus the amount eliminated; **change in alcohol in one hour** $= 14 - \frac{10a}{4.2 + a}$, where a is the amount of alcohol already in the body at the beginning of the hour.

We could deal with any drinking rate by using the formula

change in alcohol in one hour $= d - \dfrac{10a}{4.2 + a}$,

where d is the number of grams of alcohol consumed each hour and a is the amount of alcohol in the body at the beginning of an hour. While a person is periodically ingesting some amount of a chemical, such as alcohol or caffeine, the body is also eliminating some amount. After some period of time, the amount of the chemical in the body may be at an **equilibrium state**, meaning that the amount being eliminated equals the amount being ingested, or that the **change** in the amount of alcohol in the body is zero. Thus, the amount of alcohol in the body would be at equilibrium if $0 = d - \dfrac{10a}{4.2 + a}$.

You Try It # 7

Solve for a as a function of d, the number of grams of alcohol being consumed each hour. Explain what this function means.

For the following You Try Its, you might use the information that a person weighing 150 pounds (1) usually first feels the effects of alcohol when he or she has 10 grams in the body, (2) is considered intoxicated with 34 grams of alcohol, and (3) is at risk of dying with 210 grams of alcohol.

You Try It # 8

Find the equilibrium amount of alcohol in the body when a person drinks 7 grams of alcohol each hour. What can you say about this person's level of intoxication over an extended period of time if this person weighs 150 pounds?

You Try It # 9

Find the amount a of alcohol in the body at equilibrium when a person drinks 9, 9.5, and 9.9 grams of alcohol each hour. What can you say about the level of intoxication over an extended period of time for each of these three drinking rates?

You Try It # 10

Graph the function for a in terms of d. Where is its vertical asymptote and what does it say about a person's level of intoxication?

Your graph should show that for d greater than the vertical asymptote, the value of a is negative. There is no positive equilibrium value if a person drinks more than a certain amount each hour. Instead, the amount of alcohol in this person's system would continue to increase until she or he stopped drinking or went into a coma. Discuss this result in relation to the fact that the standard drink (one beer, one glass of wine, or one mixed drink) contains 14 grams of alcohol.

Caffeine and Alcohol: Different Biochemistry, Different Mathematics

If a constant amount of caffeine is ingested into the body each hour, regardless of what that amount is, the amount of caffeine in the bloodstream will approach equilibrium. We could model caffeine's presence in the body on the calculator by inputting some number C, then repeating

C + 0.87Ans

many times. Try it. Can you see from this what equilibrium means? This model does not work for alcohol.

Modeling the processes of the human body with mathematics helps medical scientists investigate more effectively.

You Try It # 11

From the formula, compute the number of grams of alcohol consumed per hour that achieves an equilibrium level of alcohol of:

a) $a = 140$ grams.

b) $a = 210$ grams. (At this level, there is risk of death for a person weighing 150 pounds.)

From You Try It # 11, you see that the difference between the hourly consumption that achieves a high level of intoxication and the amount that puts this 150-pound person at risk of death is about one tenth of one gram. This means that when the consumption level d is near the vertical asymptote, small changes in d result in **large** changes in the equilibrium amount in the body. Variability in sizes of drinks and variability in the human body cause the numbers 10 and 4.2 in the formula to be only approximate, so it is impossible to consume a fixed amount of alcohol per hour to achieve an equilibrium level of alcohol in the body that maintains a moderate to high level of intoxication. The following You Try It gives another demonstration of this fact.

You Try It # 12

Someone, with little understanding of vertical asymtotes, decides to try to achieve an equilibrium level of 80 grams of alcohol in the body. He uses the formula $d = \dfrac{10a}{4.2 + a}$ to find that he should consume $d = 9.5$ grams of alcohol per hour. Recall that this formula may vary from person to person. For example, the 10 in the numerator might be 9, 11, or some other number reasonably close to 10. Suppose, in reality, the formula for this person is $d = \dfrac{9.6a}{4.2 + a}$ instead of $d = \dfrac{10a}{4.2 + a}$.
What is the true equilibrium level of alcohol in this person's body, given the consumption rate of $d = 9.5$ grams per hour? Is this person's life at risk?

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Some Answers to the You Try Its

The section entitled **The Process of Elimination** deals with the study of horizontal asymptotes.

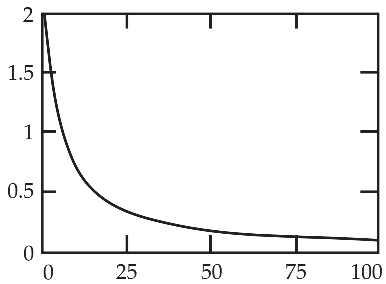
1 a $p = \frac{10}{4.2+14} \approx 0.549.$

b $p = \frac{10}{4.2+28} \approx 0.311.$

c $p = \frac{10}{4.2+42} \approx 0.216.$

2 Solve $1 = \frac{10}{4.2+a}$; $a = 5.8$.

3 The graph of $p = \frac{10}{4.2+a}$ is given below. $p < 0.1$ when $a > 95.8$.



4 $y = \frac{10 \times 14}{4.2+14} \approx 7.69$; $y = \frac{10 \times 28}{4.2+28} \approx 8.69$; $y = \frac{10 \times 42}{4.2+42} \approx 9.09.$

Comparison of You Try It # 1 with You Try It # 4 reveals that as the amount of alcohol a increases, the amount being eliminated increases (You Try It # 4), but the proportion of the alcohol present that is being eliminated decreases (You Try It # 1). Ascertain that students see the significantly different way the two functions act as a increases and what this means physically.

5 Horizontal asymptote at $y = 10$. This tells us that as the amount of alcohol in the body increases, the amount eliminated approaches a ceiling of 10 grams per hour.

6 For $a \in [34,160]$, the function value is near its asymptote $y = 10$. We can use 10 as an approximation of the number of grams eliminated per hour. Since 160 – 34 = 126 grams need to be removed from the body, 126/10 = 12.6 or 13 hours estimates the time needed. 12.6 is an underestimate since 10 is an overestimate of the number of grams eliminated per hour, which can be seen from observing the graph of $y = \frac{10a}{4.2+a}$.

Pause here to reflect on (1) what a horizontal asymptote is, (2) what the horizontal asymptote to $p = \frac{10a}{4.2+a}$ means in the alcohol context, and (3) how we might have recognized from the equation that the end behavior of this function would be an approach to $p = 10$ (that is, how we might easily recognize what the horizontal asymptote of a rational function is).

The section entitled **Equilibrium** deals with the study of vertical asymptotes.

7 $a = \frac{4.2d}{10-d}$. This function gives the amount of alcohol in the body of a person who drinks d grams of alcohol every hour over a relatively long period. It is the amount of alcohol in the body when it reaches its equilibrium state for this size drink, where continued ingestion at this rate maintains a body level of grams of alcohol.

8 $\frac{4.2 \times 7}{10-7} = 9.8$. A 150-pound person may have no physical response to this rate of alcohol intake, even over a long period, since the amount in the body remains below 10g.

9 When $d = 9$, the equilibrium value is $a = \frac{4.2 \times 9}{10-9} = 37.8$; this person will be intoxicated if this drinking pattern continues over a long enough period and so should not drive once near equilibrium. When $d = 9.5$, $a = \frac{4.2 \times 9.5}{10-9.5} = 79.8$; very intoxicated. When $d = 9.9$, $a = \frac{4.2 \times 9.9}{10-9.9} = 415.8$; there is risk of death at 210 grams, well before nearing equilibrium.

10 Vertical asymptote at $d = 10$. As d approaches 10 from the left, a gets large without bound. Very small increases in d produce very large increases in a.

11 a $\frac{4.2d}{10-d} = 140$ if $d \approx 9.7$.

b $\frac{4.2d}{10-d} = 210$ if $d \approx 9.8$.

Pause here to reflect on (1) what a vertical asymptote is, (2) what the vertical asymptote to $a = \frac{4.2d}{10 - d}$ means in the alcohol context, (3) how we might have recognized from the equation what the behavior of this function would be as d approaches 10.

Note in You Try It # 9 how a very small change in d produces a large enough change in a so that in this situation there is a risk of death. The physiological significance of this is quite compelling; be sure to note the **mathematical** significance in reference to the slope of the curve.

When graphing $d = \frac{10a}{4.2 + a}$ and $a = \frac{4.2d}{10 - d}$ on a calculator using x's and y's, be sure to recognize the relationship between this pair of inverse functions. Note especially that where the first function had a horizontal asymptote ($y = 10$), the second function has a vertical asymptote ($x = 10$). Reflect on why this should be so in the context of what students know about inverse functions

You Try It # 11 further reinforces the concept that small changes in the independent variable near a vertical asymptote produce large changes in the dependent variable, but from another point of view; that is, here large changes in the dependent variable are shown to correspond to small changes in the independent variable.

12 If $d = \frac{9.6a}{4.2 + a}$ and $d = 9.5$, then $a \approx 399$.

Note that here, $a = \frac{4.2d}{9.6 - d}$, so the vertical asymptote is $d = 9.6$. For $d < 9.6$ but close, a is volatile!

Summary

Following this investigation, students should: (1) know what a rational function is; (2) know what a horizontal asymptote is both geometrically and numerically and have some idea how to recognize where one will be from the algebraic representation of the function; (3) and know what a vertical asymptote is both geometrically and numerically, and have some idea how to recognize where one will be from the algebraic representation. To gain this knowledge, however, students need to do both the investigation and then reflect on the experience and focus their thinking on the mathematical models they have used.